Algebraic Smooth Occluding Contours: Supplemental material

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1 Conic degenerate patches

A quadratic Bezier patch can be written as

$$\mathbf{p}(u,v) = \mathbf{p}_{00}w^2 + \mathbf{p}_{20}u^2 + \mathbf{p}_{02}v^2 + 2\mathbf{p}_{10}uw + 2\mathbf{p}_{01}vw + 2\mathbf{p}_{11}vu,$$

with the first three control points corresponding to the vertices and the last three to the midpoints of a Bezier triangle, and $w = 1 - u - v$. Suppose the gradient at $\mathbf{p}_{00}$ is zero; then then the equation (1) and formula for the Bezier points of subpatches below yield $\mathbf{p}_{01} = \mathbf{p}_{10} = \mathbf{p}_{00}$, i.e., the three points are collapsed to one, and the patch reduces to

$$\mathbf{p}(u,v) = \mathbf{p}_{00}w(w + 2v + 2u) + \mathbf{p}_{20}u^2 + \mathbf{p}_{02}v^2 + 2\mathbf{p}_{11}vu$$

A direct computation yields the following expressions for the tangents:

$$\mathbf{p}_u = 2(\Delta\mathbf{p}_{20}u + \Delta\mathbf{p}_{11}v),\ \mathbf{p}_v = 2(\Delta\mathbf{p}_{02}v + \Delta\mathbf{p}_{11}u)$$

where $\Delta\mathbf{p}_{ij} = \mathbf{p}_{ij} - \mathbf{p}_{00}$. From these, we can see that the tangent direction for any line $au + bv = 0$ is constant, i.e., the 3D images of these lines, passing through $\mathbf{p}_{00}$, are straight lines, and the patch is a cone. For these patches, the contour lines are stably a pair of intersecting lines. We also note that if the surface is tangent plane continuous at such a point, it follows that the three vectors $\Delta\mathbf{p}_{ij}$ have to be coplanar in general, i.e., all control points of the patch are in the same plane and the patch is flat. We found that forcing such flat spots around these vertices is often less preferable compared to allowing a cone vertex, in terms of contour generator behavior.

2 Powell-Sabin interpolant

We describe the construction for a scalar function. Exactly the same construction is applied to each of x, y and z coordinate functions. For completeness, we describe the Powell-Sabin interpolant construction for the global parametrization setting in which the gradient degrees of freedom per vertex are defined on vertex charts. As explained in the text, first, for each corner of a triangle (i,j,k) transforms T_{ijk} are applied to transform the gradients g^u, g^v defined in the coordinate charts for each vertex i,j,k, and g^m defined in local coordinates on an edge chart, to the global coordinates u, v in which the parametric coordinates $\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k$ of the triangle are defined. We proceed with defining the set of quadratic patches as follows.

Given these 12 degrees of freedom transformed to the global coordinates, we first convert them to an affine-invariant form of derivatives per patch.

We switch from global triangle vertex indices (i,j,k) to local indices $(0,1,2)$. The per-triangle local degrees of freedom are: (1) 3 vertex values p_i, $i = 0,1,2$, (2) 6 derivatives of $p(u,v)$ at vertices i in the directions of parametric edges e_{ij}, which we denote d_{ij}, and h_{ij}, (3) 3 derivatives of $p(u,v)$ evaluated at midpoints m_{ij} of the edges, in the direction of the opposite vertex k, where (i,j,k) is a permutation of $(0,1,2)$.

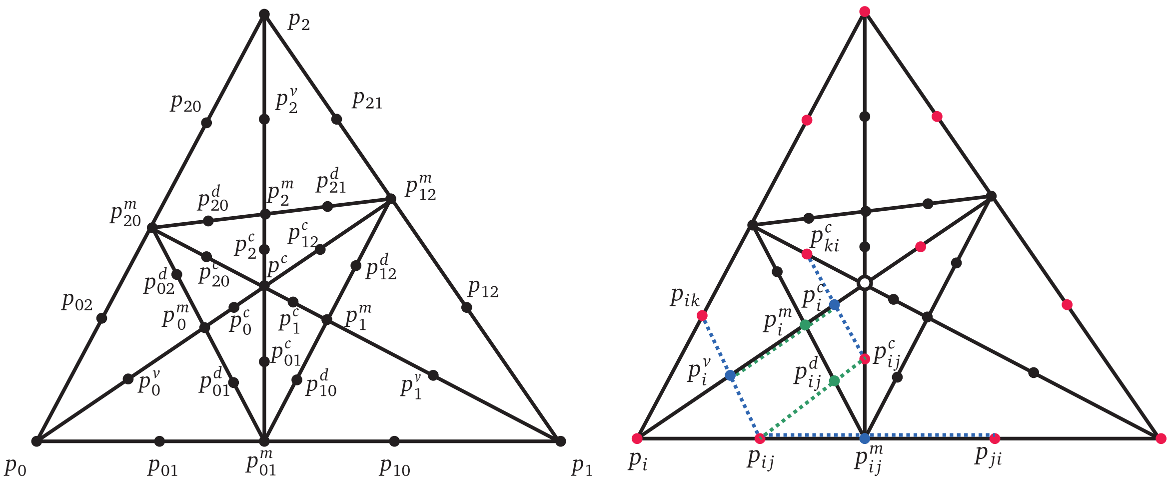

Figure 1: Left: names of control points for the Bezier representation of the patches in the 12-split Powell-Sabin interpolant. Right: red points indicate control points computed directly from the input data; blue points are computed by averaging of pairs of points connected by the blue dotted lines. green points are obtained by averaging points at the ends of green dotted lines with weights 3/4 and 1/4.

The transformed vectors (g_i^u, g_i^v) are derivatives of the quadratic interpolant in directions u and v, and g_{ij}^m are derivatives along the direction $\mathbf{e}_{ij}^{\perp}$. The derivative g_{ij}^e, along $\mathbf{e}_{ij}$ is fully determined from other degrees of freedom by the requirement that the quadratic patches inside each triangle join with C^1 continuity: $g_{ij}^e = -2p_i - d_{ij}/2 + 2p_j + d_{ji}/2$. This leads to the following linear map for the triangle local degrees of freedom (note that the coefficients of the equations depend on the parametric coordinates only, so are view-independent).

$$d_{ij} = g_i^u e_{ij}^u + g_i^v e_{ij}^v, \quad h_{ij} = g_{ij}^e \mathbf{e}_{ij}^m \cdot \hat{\mathbf{e}}_{ij} + g_{ij}^m \mathbf{e}_{ij}^m \cdot \hat{\mathbf{e}}_{ij}^{\perp} \tag{1}$$

where $\mathbf{e}_{ij}^m = \mathbf{r}_k - \frac{1}{2}(\mathbf{r}_i + \mathbf{r}_j)$ is the vector from the midpoint to the opposite vertex in the parametric domain.

Bezier points for subpatches. Finally, given the degrees of freedom above, the Bezier control points of each of the 12 quadratic patches denoted as shown in Figure 1 in a simple form: the corner control points are just p_i, adjacent six points p_{ij} are displaced proportionately to edge derivatives d_{ij}; h_{ij} determines the displacement of p_i^c from the midpoint of (p_{ij}, p_{ij}) and the rest are determined by averaging. Note that the coefficients of these expressions are constants not depending on the parametrization, so this yields a fixed part of the matrix, with the dependence on the parametrization entirely captured by (1)

$$\begin{aligned} p_i^c &= \tfrac{1}{2}(p_{ij}^c + p_{ki}^c) & & \\ p_{ij} &= p_i + \tfrac{d_{ij}}{4} & p_i^v &= \tfrac{1}{2}(p_{ij} + p_{ik}) \\ p_{ij}^m &= \tfrac{1}{2}(p_{ij} + p_{ji}) & p_i^m &= \tfrac{p_i^v}{4} + \tfrac{3p_i^c}{4} \\ p_{ij}^c &= p_{ij}^m + \tfrac{h_{ij}}{6} & p_{ij}^d &= \tfrac{p_{ij}}{4} + \tfrac{3p_{ij}^c}{4} \\ p^c &= \tfrac{1}{3}(p_{01}^c + p_{12}^c + p_{20}^c) & & \end{aligned}$$